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IMPLICATIONS OF CONSERVATION EQUATIONS
FOR THE DETERMINATION OF ABSOLUTE VELOCITIES

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ABSTRACT

The consequences of assuming that density is conserved in the problem of determining absolute velocities are investigated. Two questions are considered: (i) the constraints that the density must satisfy to be compatible with the assumed geostrophic and hydrostatic dynamics and (ii) whether and to what extent the indeterminacy in this dynamics is removed by this additional assumption.

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1. Introduction

With the dawn of ocean forecasting approaching, the need to know velocity fields for use as initial conditions in a prediction scheme is keenly felt. But direct velocity measurements are still difficult to make, and so we are forced to rely on the classical procedure which consists of inferring the velocity fields from temperature and salinity measurements. In this procedure, we are given the density field $\rho(x,y,z)$, and we must find velocity fields $u(x,y,z), v(x,y,z), w(x,y,z)$ and a pressure field $p(x,y,z)$ such that

$$fv = \rho_0^{-1} p_x \tag{1.1a}$$

$$fu = -\rho_0^{-1} p_y \tag{1.1b}$$

$$0 = -\rho^{-1} p_z - g \tag{1.1c}$$

$$u_x + v_y + w_z = 0 \tag{1.1d}$$

The notations in the above equations are standard: x, y, z are cartesian coordinates; f is the Coriolis parameter and g is the gravitational acceleration; ρ_0 is an average value of the density.

As is very well known, the integration of the hydrostatic equation from the ocean's bottom at $z = -h$ to a depth z, viz.

$$p = -g \int_{-h}^{z} \rho(x,y,z')\, dz' + \Pi(x,y) \tag{1.2}$$

introduces an unknown function of x, y denoted here by Π. This 'barotropic' component of the pressure cannot be determined from the dynamics (1.1a,b,c,d).

The indeterminacy in the pressure entails a similar indeterminacy in the velocity fields. Indeed, if we define

$$P(x,y,z) = -g \int_{-h}^{z} \rho(x,y,z')\, dz' \tag{1.3}$$

then

$$u = -(\rho_0 f)^{-1} \{P_y + \Pi_y\} \tag{1.4a}$$

$$v = \quad (\rho_0 f)^{-1} \{P_x + \Pi_x\} \tag{1.4b}$$

The vertical velocity, which is obtained by integrating the continuity equation in the vertical,

$$w = w(x,y,-h) - \int_{-h}^{z} (u_x + v_y)\, dz' \tag{1.5}$$

also contains the same indeterminacy. The search for the famous 'level of no motion' is intimately related to the determination of $\Pi(x,y)$.

Following in the footsteps of Worthington (1976), Wunsch (1978), Fiadeiro & Veronis (1982) and others, we want to examine whether the addition of extra information in the

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form of the conservation of the density field, viz.

$$u\rho_x + v\rho_y + w\rho_z = 0 \tag{1.6}$$

can determine $\Pi(x,y)$.

Mathematically, it is not clear whether Equation (1.6) is compatible with the dynamics (1.1). We shall examine this question first on the $f-$plane and then on the $\beta-$ plane. We shall show that in both cases the density field must satisfy certain constraints. If these constraints are satisfied, then for the $\beta-$plane case, but not for the $f-$plane, Π can be determined. Our analysis is carried out in the simplest case of a flat bottom ocean. However, it can be extended to the case where bottom topography exists as well as to cases where quantities other than density are conserved.

2. A mathematical result

An essential step in our analysis consists in finding a $z-$<u>independent</u> solution of an equation of the form

$$A(x,y,z)\Pi_x + B(x,y,z)\Pi_y = 1 \tag{2.1}$$

which holds in a cylindical basin $\mathbf{D} = \{(x,y,z) : (x,y) \in D, -h < z < 0\}$, where D is the horizontal cross-section. If A and B themselves are independent of z, then the existence of such solutions is not too surprising. Of greater interest is the case in which such solutions exist eventhough A and B are z dependent, i.e when

$$A_z^2 + B_z^2 \not\equiv 0 \quad \text{in} \quad \mathbf{D}. \tag{2.2}$$

Relabeling coordinates if need be, we interpret (2.2) to mean that

$$B_z \not\equiv 0 \quad \text{in} \quad \mathbf{D} \tag{2.2'}$$

. With the above conditions we have the following result.

Theorem: There exists a z-independent solution $\Pi(x,y)$ of (2.1) if and only if for all $(x,y,z) \in \mathbf{D}$

$$\text{(i)} \quad A(x,y,z) = k_0(x,y)B(x,y,z) + l_0(x,y) \tag{2.3a}$$

$$\text{(ii)} \quad l_0(x,y) \neq 0 \tag{2.3b}$$

$$\text{(iii)} \quad \partial_y(l_0^{-1}) - \partial_x(k_0/l_0) = 0 \tag{2.3c}$$

Furthermore, this solution $\Pi(x,y)$ satisfies

$$\Pi_x = l_0^{-1}(x,y) \tag{2.4a}$$

$$\Pi_y = -k_0(x,y)/l_0(x,y) \tag{2.4b}$$

Proof: Suppose (2.3a,b,c) hold in **D**. We must show that a z-independent solution $\Pi(x,y)$ of (2.1) exists, and that it satisfies (2.4a,b).

In view of (2.3b), we can write (2.3a) thus:

$$A \, l_0^{-1} - B k_0 \, l_0^{-1} = 1 \tag{2.5}$$

Now, (2.3c) implies the existence of a function, say $\Pi(x,y)$, such that (2.4a,b) hold. Substituting l_0^{-1} and $-k_0 l_0^{-1}$ by their expressions in terms of Π in (2.4), we conclude that this function Π is a solution of (2.1).

Conversely, suppose that Π is a z-independent solution of (2.1) and that (2.2') holds. Then we can differentiate (2.1) with respect to z and write

$$A_z \Pi_x + B_z \Pi_y = 0. \tag{2.6}$$

We note at this stage that $A_z B - B_z A \not\equiv 0$ in **D**. Indeed, if this were the case, then

$$B_z(A\Pi_x + B\Pi_y) = B_z.$$

which is obtained by multiplying (2.1) by B_z, could be written as

$$B(A_z \Pi_x + B_z \Pi_y) = B_z.$$

But as a result of (2.6), we reach a contradiction. Therefore

$$A_z B - B_z A \not\equiv 0 \tag{2.7}$$

and as a result, we can solve (2.1) and (2.6) for Π_x and Π_y to find

$$\Pi_x = \frac{B_z}{AB_z - BA_z} \tag{2.8a}$$

$$\Pi_y = -\frac{A_z}{AB_z - BA_z} \tag{2.8b}$$

This step is not unlike that taken by Stommel & Schott (1977) and Needler (1985) in their study of this problem.

We must next insure that these expressions for Π are indeed z-independent. By considering their ratio, we immediatedly see that

$$A_z = k_0(x,y)B_z$$

and after integration we arrive at the condition (2.3a) between A and B. Substituting this condition in (2.8a,b), we arrive at (2.4a,b). Condition (2.3c) arises from forming Π_{xy} in

two different ways. Finally, the condition (2.7) translates into (2.3b). We can easily verify that further z differentiations do not introduce other conditions. This completes the proof.

In closing this section, we give an example of a partial differential equation of the form (2.1) in which the coefficients A and B satisfy all the conditions of the theorem. The example is:

$$[B(x,y,z) + \frac{1}{f'(x-y)}]\Pi_x + B(x,y,z)\Pi_y = 1$$

where B is an analytic function of z. In this case,

$$k_0 = 1$$

$$l_0 = \frac{1}{f'(x-y)}$$

and

$$\Pi = f(x-y)$$

3. The f-plane

We consider in this section the case of the f-plane, simply as a foil for the β-plane which is discussed next. In this case, the flow is horizontally non-divergent. This, together with the fact that the bottom is flat, implies that

$$w = 0 \tag{3.1}$$

Thus, the added conservation equation reads

$$u\rho_x + v\rho_y = 0 \tag{3.2}$$

Making use of the hydrostatic and geostrophic balances, this equation can be written as follows:

$$-p_y p_{xz} + p_x p_{yz} = 0 \tag{3.3}$$

and since $p_x^2 + p_y^2$ is not identically zero in $\mathbf{D}$ lest the ocean would be at rest, this implies that

$$\partial_z \left(\frac{p_x}{p_y} \right) = 0$$

i.e.

$$p_x = -p_y \tan \alpha_0(x,y) \tag{3.4}$$

or equivalently

$$v(x,y,z) = u(x,y,z)\tan \alpha_0(x,y) \tag{3.5}$$

Thus, at a given latitude and longitude, the velocity, which is purely horizontal, has the same direction for all depths! This result, in turn, places some constraint on the data. Indeed. substituting (3.5) in (3.2) shows that

$$\rho_x + \rho_y \tan \alpha_0 = 0 \tag{3.6}$$

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This constraint is typical of the constraints the conserved quantity must satisfy for the data to be compatible with the assumed dynamics. What this means in practice, is that at each station, z-independent α_0 are generated by passing the best fitting line passing through points with absiscas ρ_y and ρ_x (or whatever the corresponding quantities are for the case at hand). Figure 1 shows an attempt to determine one such angle α_0 from data from the REX (Regional Energetics Experiment) taken on August 13, 14 and 15 1985 at a station in the North Atlantic located at latitude 37^0 N and longitude 64^0 E. The density was obtained from AXBT temperatures measured every two meters up to a depth of 743 meters merged with climatological salinity data. Figures 2(a), (b) and (c)*show the same data segregated by depth.

As we shall see, a similar constraint arises on the β-plane.Having satisfied this constraint, are we now capable to determine Π? The answer is unfortunately no. Indeed, from the definition (1.3) of P. it follows that:

$$P_x + P_y \tan \alpha_0 = 0 \tag{3.7}$$

This means that the baroclinic components of the flow field satisfy the conservation equation by themselves. Therefore, the need for a barotropic correction is not imperative. From (3.4) and (3.7) it follows that

$$\Pi_x + \Pi_y \tan \alpha_0 = 0 \tag{3.8}$$

If we were forced to proceed further, then we would need to know the distribution of sources and sinks at the edge of our basin in order to solve this equation for Π.

4. The β-plane

Because of the β-effect. the horizontal mass divergence is $-\beta v/f_0$ and therefore the vertical velocity as given by (1.5) is

$$w(x,y,z) = \beta f_0^{-1} \int_{-h}^{z} v(x,y,z')\,dz' \tag{4.1}$$

or in terms of P and Π

$$w = (\beta/f_0^2 \rho_0)\{\int_{-h}^{z} P_x\,dz' + [z + h]\,\Pi_x\} \tag{4.2}$$

We shall find it convenient at times to write

$$W = (\beta/f_0^2 \rho_0) \int_{-h}^{z} P_x\,dz' \tag{4.3}$$

The conservation equation for ρ now becomes

$$\{\rho_y + (\beta[z + h]/f_0)\rho_x\}\Pi_x - \rho_x \Pi_y = -\rho_y P_x + \rho_x P_y - (\rho_0 f_0)\rho_z W \tag{4.4}$$

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* We are indebted to Dr. Ted Bennett for providing us with these figures.

The question once again is whether the right hand side of (4.4) vanishes, i.e. whether the baroclinic fields identically satisfy the conservation equation. We recall that for $\beta = 0$ the answer to this question was always yes. If each side of (4.4) is separately zero, then the density field which makes up the data is constrained to satisfy two distinct equations. The equation stemming from the baroclinic part is

$$-\rho_y P_z + \rho_z P_y - (\rho_0 f_0)\rho_z W = 0 \tag{4.5}$$

whereas that from the barotropic part yields

$$\rho_x + \{\rho_y + (\beta[z + h]/f_0)\rho_z\} \tan \alpha_0 = 0 \tag{4.6}$$

where α_0 is, as previously, solely a function of x and y. Are there isopycnal surfaces which satisfy both of the above equations?

We shall show that this question must be answered negatively. Integrating (4.6) over z and using the definition (1.3) of P. we deduce that

$$P_x + \tan \alpha_0 \{P_y + \frac{\beta(z + h)}{f_0}\rho g + \frac{\beta}{f_0} P\} = 0 \tag{4.7}$$

Next, we eliminate ρ_y and P_y from (4.5) using (4.6) and (4.7). The result is:

$$\beta f_0^{-1} \rho_x \{P + \rho g \int_{-h}^{z} dz'\} = \beta f_0^{-1} \rho_z \{-P_x \int_{-h}^{z} dz' + \int_{-h}^{z} P_x dz'\} \tag{4.8}$$

If we differentiate this equation with respect to z, we see that

$$\beta f_0^{-1} \rho_{xz} \{P + \rho g \int_{-h}^{z} dz'\} = \beta f_0^{-1} \rho_{zz} \{-P_x \int_{-h}^{z} dz' + \int_{-h}^{z} P_x dz'\} \tag{4.9}$$

The above two equations imply that

$$\rho_z \rho_{xz} = \rho_x \rho_{zz}$$

i.e.

$$\rho_x = \rho_z \tan \gamma_0(x, y) \tag{4.10}$$

We have thus reduced the problem to finding whether there exists a function $\rho(x, y. z)$ which satisfies the system of equations

$$E_1[\rho] \equiv \tan \gamma_0 \rho_x - \rho_z = 0$$
$$E_2[\rho] \equiv \rho_x + \tan \alpha_0 \rho_y + (\beta[z + h]/f_0) \tan \alpha_0 \rho_z = 0$$

At this stage we appeal to the theory of systems of first order partial differential equations (see e.g. Smirnov 1964, p.34.). The first step in the theory consists in attempting to derive further independent equations for ρ. Such additional equations are obtained by forming

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Poisson brackets. Thus, denoting temporarily x,y,z by x_1, x_2, x_3 and ρ_x, ρ_y, ρ_z by p_1, p_2, p_3, the theory leads us to consider

$$E_3[\rho] \equiv \sum_{i=1}^{3} \{ (\partial E_1/\partial x_i)(\partial E_2/\partial p_i) - (\partial E_2/\partial x_i)(\partial E_1/\partial p_i) \} = 0$$

i.e.

$$-[(\partial\gamma_0/\partial x) + (\partial\gamma_0/\partial y)\tan\alpha_0)(\cos\gamma_0)^{-2}]\rho_x + [(\partial\alpha_0/\partial y)\tan\gamma_0(\cos\alpha_0)^{-2}]\rho_y$$
$$+ [\beta(z+h)f^{-1}(\partial\alpha_0/\partial y)\tan\gamma_0(\cos\alpha_0)^{-2} - \beta f^{-1}\tan\alpha_0]\rho_z = 0 \quad (4.11)$$

This equation is independent of the other two. Having obtained <u>three</u> independent, homogeneous, first order partial differential equation for ρ which is a function of three independent variables, the theory tells us that $\rho = $ constant is the only possible solution.

Thus, on the β-plane the baroclinic velocity fields cannot satisfy the density conservation. Therefore, we can write (4.4) as

$$A\Pi_x + B\Pi_y = 1 \qquad (4.12)$$

where

$$A = \frac{\rho_x + \rho_y[\beta(z+h)/f_0]}{\rho_x P_y - \rho_y P_x - \rho_z W} , \qquad (4.13a)$$

and

$$B = -\frac{\rho_x}{\rho_x P_y - \rho_y P_x - \rho_z W} . \qquad (4.13b)$$

If we assume that A and B are functions of depth, then the theorem yields the following constraint on the data

$$A = k_0(x,y)B + l_0(x,y)$$

and the usual expressions for the barotropic components, namely

$$\Pi_x = l_0^{-1}(x,y)$$
$$\Pi_y = -k_0(x,y)/l_0(x,y)$$

On the β-plane, the density conservation is stringent enough to determine the barotropic field completely.

5. Concluding remarks

We have attempted to elucidate the question of how a depth dependent conserved field can remove the depth independent indeterminacy of the velocity fields. We have seen that this attempt leads naturally to a characterization of the class of acceptable density fields. In turn, this suggests a very natural variational procedure, namely that the "distance"

between the data and this set of acceptable densities be mimimized. For the β–plane, there is a unique velocity field associated with this projection. Mr. T. Bennett of NORDA is currently implementing this procedure on data from the North Atlantic.

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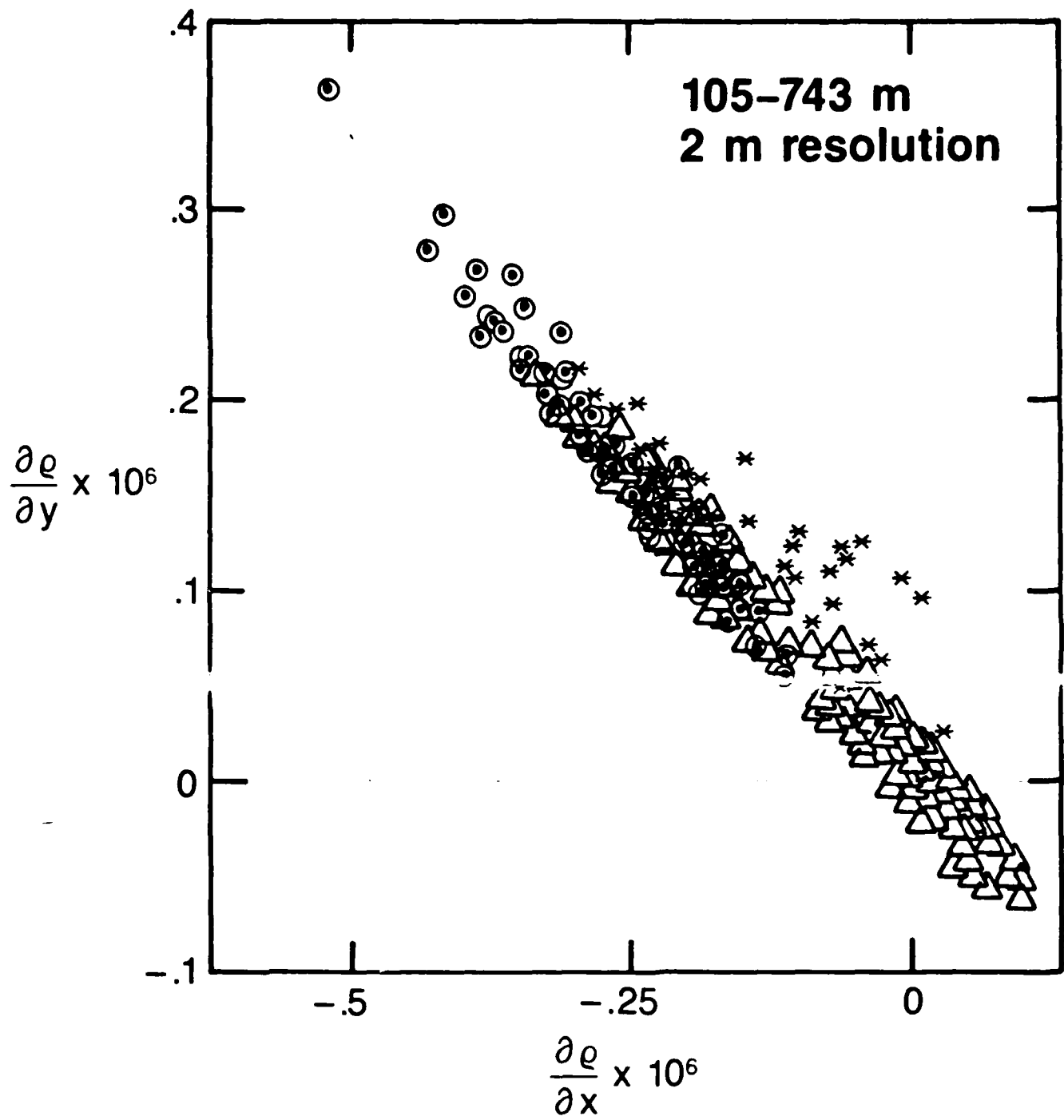
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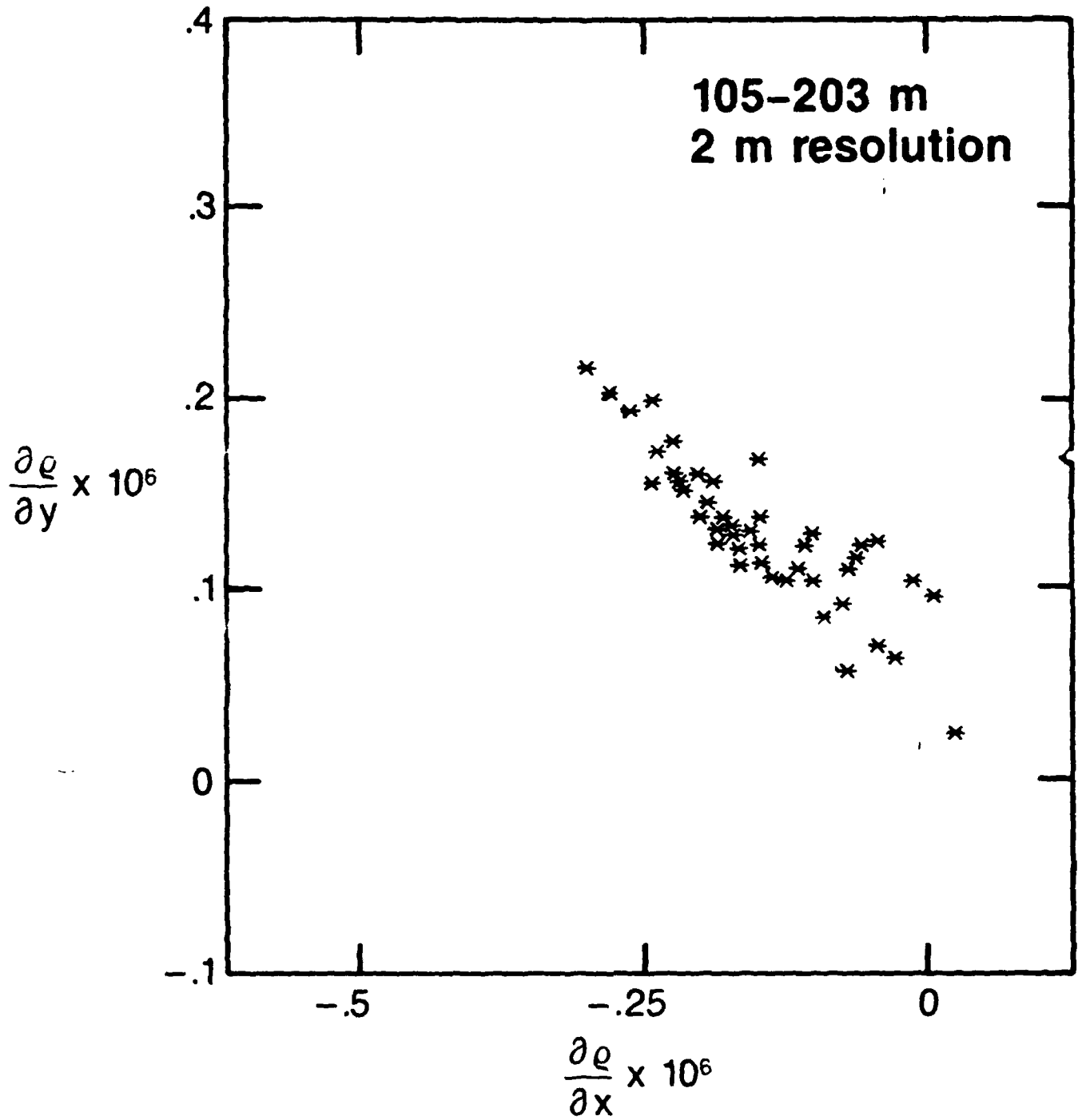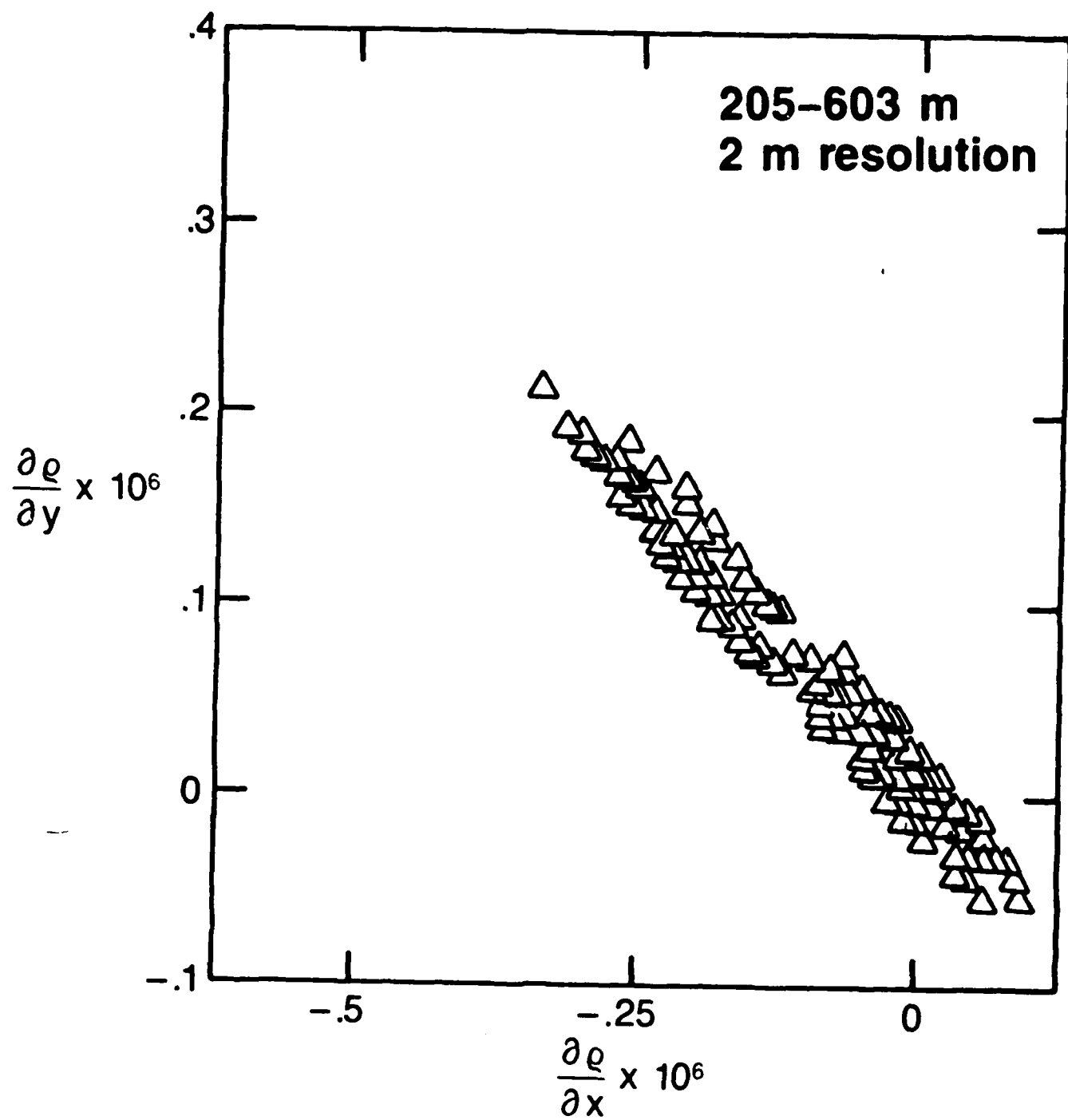
Figure Captions

Figure 1. N-S density gradient vs. E-W density gradient at 37^0 N and 64^0 E for different depths. The measurements were taken on August 13-15, 1985 as part of the REX experiment. The stars correspond to measurements over a depth from 105m to 203m. Open triangles correspond to depths from 205m to 603m. Finally, dotted circles correspond to depths from 605m to 743m. A stright line passing through the origin with a slope of -.7 provides a good fit.
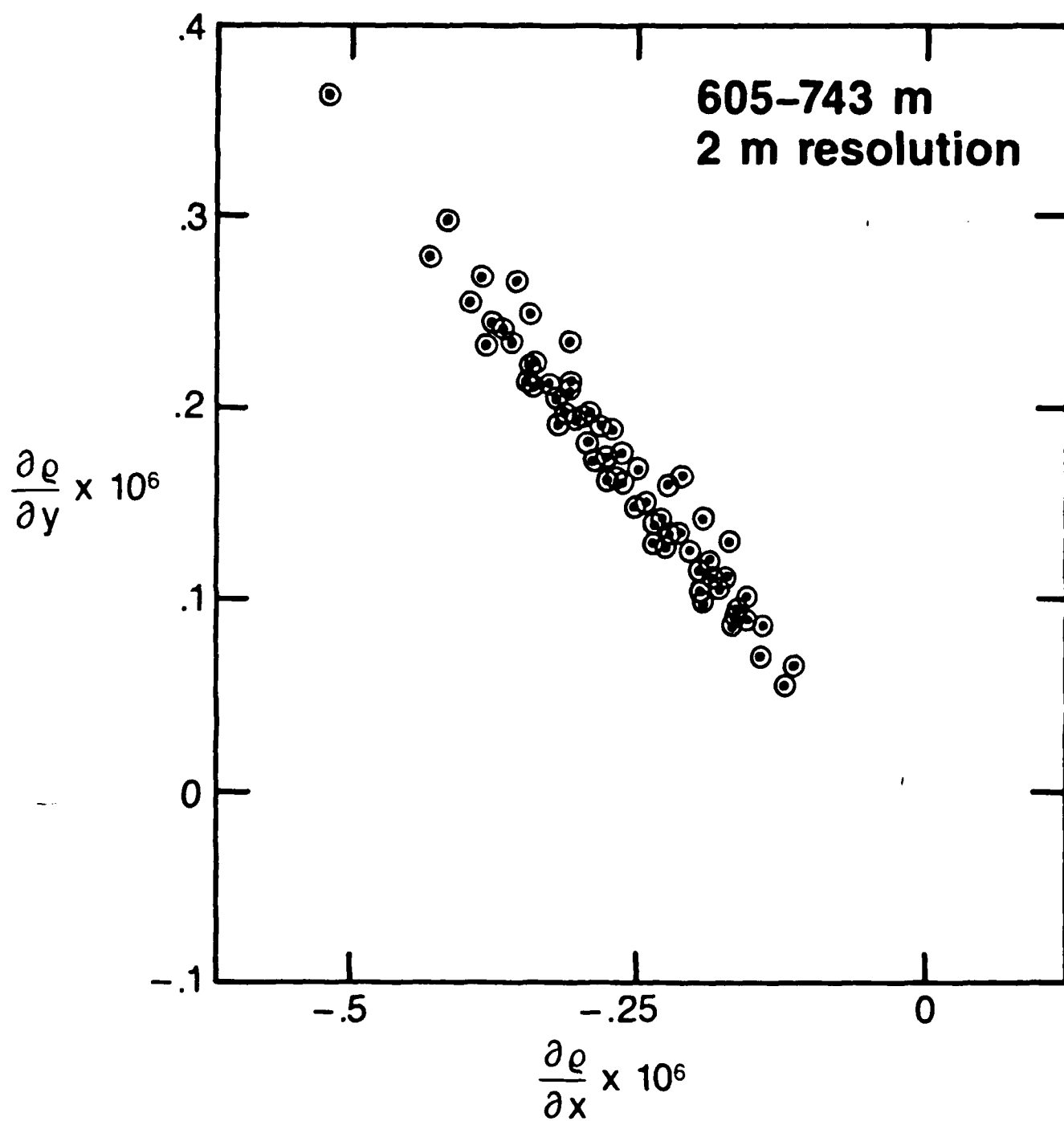
Figures 2. Same data as in Figure 1 segregated by depth. (a) Upper layer (105m to 203m). Note the large scatter. (b) Intermediate layer (205m to 603m). (c) Bottom layer (605m to 743m)

105–743 m
2 m resolution

$\dfrac{\partial \varrho}{\partial y} \times 10^6$

$\dfrac{\partial \varrho}{\partial x} \times 10^6$

$\times \sim 2.53$

105–203 m
2 m resolution

$\dfrac{\partial \varrho}{\partial y} \times 10^6$

$\dfrac{\partial \varrho}{\partial x} \times 10^6$

605–743 m
2 m resolution

$\frac{\partial \varrho}{\partial y} \times 10^6$

$\frac{\partial \varrho}{\partial x} \times 10^6$

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